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Non-quasi-convex Problems in Nonlinear Elastostatics*

L. T. CAMPOS AND J. T. ODEN

Aerospace Engineering Department, University of Texas, Austin, Texas 78712

1. INTRODUCTION

This paper describes a study of a class of non-quasi-convex problems in finite elastostatics. The problems of interest here are equilibrium problems in which equilibrium states are characterized as minimizers of a potential energy functional. For non-quasi-convex problems, these minimization problems may not always have a solution. Nevertheless, we associate with each problem a so-called relaxed problem, its quasi-convexification, and give conditions under which the relaxed problem is equivalent (in a sense to be specified later) to the original one. For the relaxed problem, a solution always exists, and although it may not be a solution of the original problem, we establish in what sense it may be considered a generalized solution of it.

To provide a brief review of the collection of ideas leading up to the present study, we note that in a fundamental paper published over a quarter of a century ago, Morrey [18] introduced a notion of quasi-convexivity of integrands of certain nonconvex functionals encountered in the calculus of variations. For example, consider the problem of seeking a vector-valued function $\mathbf{u}$ ($\mathbf{u}(\mathbf{x}) \in \mathbf{R}^N$) which provides a global minimizer of multiple integrals of the type

$$I(\mathbf{v}) = \int_\Omega f(\nabla \mathbf{v})\, dx,$$

where Ω is an open bounded domain in $\mathbf{R}^N$, $I(\cdot)$ is a nonconvex functional on a suitable class of admissible functions $\mathbf{v}$ defined on Ω, and f is a

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continuous real-valued function of the gradient $\nabla\mathbf{v}$ of $\mathbf{v}$. The integrand f is said to be quasi-convex at a point $\mathbf{u}$ if and only if[1]

$$\int_D f(\nabla(\mathbf{u}+\boldsymbol{\zeta})\,dx \geqslant \operatorname{meas}(D) f(\nabla\mathbf{u})$$

for all bounded sets $D \subset \mathbb{R}^N$ and all smooth vector-valued functions $\boldsymbol{\zeta} \in (C_0^\infty(D))^N$.

The significance of quasi-convexity is that, together with certain mild growth conditions, it is a necessary and sufficient condition for the weak lower semicontinuity of functionals like $I(\cdot)$ defined on subsets of Sobolev spaces. Thus, such minimization problems with integrands which are not quasi-convex generally have no solutions.

The vital role of quasi-convexity in this class of nonconvex optimization problems has generated much interest in the field of finite elasticity since important equilibrium problems in elasticity can be formulated as problems of minimizing the total potential energy. Indeed, the "main open problem" in the general theory of elasticity is, according to Truesdell [26], to establish conditions on the stored energy functions sufficient to guarantee the existence of physically reasonable solutions to the equations of elasticity. Thus, it would seem that a reasonable theory of hyperelasticity would involve energies which could be minimized, and this, in turn, suggests that the stored energy function be quasi-convex. This observation led Ball [2, 3] to regard quasi-convexity as a constitutive assumption for nonlinear elastic bodies. Specifically, Ball introduced the concept of *polyconvexity* (to be defined later) as a more natural property for integrands encountered in finite elasticity, and he showed that polyconvexity of the stored energy function implies both quasi-convexity and that the strong ellipticity condition (the Legendere–Hadamard condition) is satisfied, if sufficient smoothness is assumed. Thus, with polyconvexity as a constitutive assumption, it is possible to develop theorems on the existence of solutions to equilibrium problems in finite elasticity.

Nevertheless, there has been an increasing number of interesting examples of elasticity problems described in the literature in which quasi-convexity does not hold. We mention in this regard the problem studied by Sternberg and Knowles of materials in which, in certain states of homogeneous strain, the strong ellipticity condition is violated. In such problems, discontinuities in strains can occur, the equilibrium equations become hyperbolic in certain domains, elastostatic shocks representing surfaces of discontinuity of certain strain measures are formed, and a variety of

[1] We give a more general and precise definition of quasi-convexity in Section 2 which follows that of Meyers [17].

characteristics of the solution unfold which are quite unusual for problems in elasticity theory. There has also been the dramatic examples of Ericksen [12] on materials with phase changes and the parallel work of James [14, 15], Dunn [7, 8], and others where materials are described which are characterized by what are easily shown to be non-quasi-convex stored energy functions. In these cases, global minimizers of the total energy function do not, in general, exist.

A useful idea used in the study of nonconvex optimization problems is that of relaxation. Ekeland and Temam [9] have shown that certain nonconvex problems can be replaced in a natural way by a regularized problem which is convex and solvable. These ideas were used by Gurtin and Temam [13] to study a nonconvex problem connected with finite antiplane shear of a hyperelastic cylinder. Recently, Dacorogna [4] made a substantial generalization of this collection of ideas by introducing the idea of quasi-convexification of a given functional defined by a non-quasi-convex integrand. This idea provides for the construction of a relaxation procedure which produces a related quasi-convex functional, but the relaxed functional may be nonconvex. There are, of course, other forms of regularized functionals which may have bearing on non-quasi-convex problems.

From its inception, it has been recognized that the property of quasi-convexity, not being local in character, is extremely difficult to verify for given stored energy functionals. Polyconvexity, on the other hand, is a more natural property for many of the stored energy functions known in finite elasticity. In the present paper, we give examples of existence theorems proving the existence of generalized solutions to certain non-quasi-convex problems in finite elasticity.

2. Non-quasi-convex Problems in Nonlinear Elastostatics

In what follows, we use standard concepts and notation from functional analysis and Sobolev spaces. For additional details, standard sources such as Yosida [28] and Adams [1] can be consulted.

2.1. *Quasi-convexity and Its Relationship to Weak Lower Semicontinuity*

The notion of quasi-convexity that we will use here was introduced by Morrey [18–20], and used by Meyers [17] and Ball [2].

Let Ω denote an open bounded connected set in $\mathbb{R}^N$ and let $\mathbf{u}(\mathbf{x}) \equiv (\mathbf{u}_1(\mathbf{x}), \mathbf{u}_2(\mathbf{x}), \ldots, \mathbf{u}_M(\mathbf{x}))$ denote a function defined on Ω and which assumes values in $\mathbb{R}^M$. For $l = 0, 1, \ldots,$ let $D^l\mathbf{u}$ denote the tensor-valued function whose components are all the components of the generalized αth derivative of $\mathbf{u}$ for $|\boldsymbol{\alpha}| \leqslant l$. Continuing, let $f_1 = f_1(\mathbf{x}, D^l\mathbf{u})$ be a continuous real-valued integrand defined for all values of $D^l\mathbf{u}$, and let for $\mathbf{u}(\mathbf{x}) \in (W^{l,\infty}(\Omega))^M$,

$$I(\mathbf{u}, \Omega) = \int_\Omega f_1(\mathbf{x}, \mathbf{D}^{l-1}\mathbf{u}, D^l\mathbf{u})\, dx, \tag{2.1}$$

where we used the notation $f_1(\mathbf{x}, D^0\mathbf{u}, D^1\mathbf{u}, \ldots, D^{l-1}\mathbf{u}, D\mathbf{u}) \equiv f_1(\mathbf{x}, \mathbf{D}^{l-1}\mathbf{u}, D^l\mathbf{u})$. Then we say that f_1 is *quasi-convex* in $(D^l\mathbf{u})$ if and only if

$$\int_\Omega f_1\big(\mathbf{x}, \mathbf{D}^{l-1}\mathbf{w}(\mathbf{x}), D^l\mathbf{w}(\mathbf{x}) + D^l\mathbf{z}(\mathbf{y})\big)\, dy \geqslant \int_\Omega f_1\big(\mathbf{x}, D^l\mathbf{w}(\mathbf{x})\big)\, dy \tag{2.2}$$

for every polynomial $\mathbf{w}(\mathbf{x})$ of degree $\leqslant l$, every $\mathbf{z} \in (W_0^{l,\infty}(\Omega))^M$ and every set $\Omega \subset \mathbb{R}^N$.

It can be shown (Morrey [18, p. 43], Meyers [17, p. 128]) that in verifying quasi-convexity, it is sufficient to consider any open bounded domain Ω, and to verify (2.2) for these $\mathbf{z} \in (C_0^\infty(\Omega))^M$. Moreover, since $D^l\mathbf{w}$ is an arbitrary constant vector $\mathbf{c}$, (2.2) can take the form:

$$\int_\Omega f_1\big(\mathbf{x}, \mathbf{c} + D^l\mathbf{z}(\mathbf{y})\big)\, dy \geqslant f_1(\mathbf{x}, \mathbf{c}) \operatorname{meas} \Omega. \tag{2.3}$$

The usefulness of considering quasi-convexity as a constitutive assumption in nonlinear elastostatics was illustrated by Ball [2, 3] and relies on the fact that (in an appropriate space) quasi-convexity of f_1 is basically a necessary and sufficient condition for weak lower semicontinuity of I. Specifically, we shall record below a basic theorem, due to Meyers [17, p. 141–142], which generalizes a result of Morrey [19] and which is of great importance. Let us first introduce the idea of a class $\mathfrak{T}_r(\Omega)$ of functions.

For any bounded domain $\Omega \subset \mathbb{R}^N$, we shall say that

$$f_1 \to f_1(\mathbf{x}, \mathbf{D}^l\mathbf{u}),$$

a continuous integrand, is of class $\mathfrak{T}_r(\Omega)$ $(1 \leqslant r < \infty)$ if and only if:

(a) $\exists c_1 \in \mathbb{R}: f_1(\mathbf{x}, \mathbf{D}^l\mathbf{u}) \leqslant c_1\{1 + |\mathbf{D}^l\mathbf{u}|\}^r$.

(b) $\exists c_2 \in \mathbb{R}$ and $\exists 0 < \gamma \leqslant 1$:

$$|f_1(\mathbf{x}, \mathbf{D}^l\mathbf{u} + \mathbf{D}^l\mathbf{v}) - f_1(\mathbf{x}, \mathbf{D}^l\mathbf{u})| \leqslant c_2\{1 + |\mathbf{D}^l\mathbf{u}| + |\mathbf{D}^l\mathbf{v}|\}^{r-\gamma}|\mathbf{D}^l\mathbf{v}|^\gamma$$

(c) There exists a continuous increasing function z, with $z(0) = 0$, such that:

$$|f_1(\mathbf{x} + \mathbf{y}, \mathbf{D}^l\mathbf{u}) - f_1(\mathbf{x}, \mathbf{D}^l\mathbf{u})| \leqslant \{1 + |\mathbf{D}^l\mathbf{u}|\}^r \cdot z\{|\mathbf{y}|\}, \tag{2.4}$$

where a, b, and c are valid for all values of the arguments.

Then we have:

THEOREM 2.1. *Let Ω be a open bounded connected set in $\mathbf{R}^N$, and let $f_1(\mathbf{x}, \mathbf{D}^l\mathbf{u})$ be a continuous integrand of class $\mathfrak{I}_r(\Omega)$ $(1 \leqslant r < \infty)$. Let $I(\mathbf{u}, \Omega)$ be defined as in* (2.1), *and denote by D either a Dirichlet class in $(W^{l,r}(\Omega))^M$ or the full space $(W^{l,r}(\Omega))^M$ itself. Then $I|_D$ is weakly sequentially lower semicontinuous in $(W^{l,r}(\Omega))^M$ if and only if*:

(i) $f_1(\mathbf{x}, \mathbf{D}^{l-1}\mathbf{u}, D^l\mathbf{u})$ is quasi-convex in $D^l\mathbf{u}$ for each fixed value of $(\mathbf{s}, \mathbf{D}^{l-1}\mathbf{u}(\mathbf{s}))$.

(ii) $\liminf_{k\to\infty} I(\mathbf{u}_k, \Omega') \geqslant -\mu(\operatorname{meas}\Omega')$ (2.5)

for every subdomain Ω' and every sequence $\mathbf{u}_k$ in D such that $\mathbf{u}_k \equiv \mathbf{u}$ on $\Omega - \Omega'$ and $\mathbf{u}_k \rightharpoonup \mathbf{u}$ in $(W^{l,r}(\Omega))^M$, where μ is a continuous increasing function with $\mu(0) = 0$ depending only on $\mathbf{u}$ and on $\limsup_{k\to\infty} \|\mathbf{u}_k\|_{(W^{l,r}(\Omega))^M}$. □

2.2. *Quasi-convexification*

We define the *quasi-convexification* Qf_2 of f_2 by

$$Qf_2 = \sup_p \{ p \leqslant f_2, p \text{ is quasi-convex integrand in } D^l\mathbf{u} \}. \tag{2.6}$$

We then define the functional $QI(\mathbf{u}, \Omega)$ as

$$QI(\mathbf{u}, \Omega) = \int_\Omega Qf_2(\mathbf{x}, \mathbf{D}^l\mathbf{u})\, dx. \tag{2.7}$$

In nonlinear elastostatics, we are interested in the case $l = 1$. The following result which follows from Dacorogna [4] is of special interest.

PROPOSITION 2.2. *Let f_2 be a continuous integrand of the form*

$$f_2(\mathbf{x}, \mathbf{u}, \nabla\mathbf{u}) = f_3(\mathbf{x}, \nabla\mathbf{u}) + f_4(\mathbf{x}, \mathbf{u}), \qquad \forall\, \mathbf{u} \in \left(W^{1,p}(\Omega)\right)^m \tag{2.8}$$

where f_3 and f_4 are both continuous in their arguments and where f_4 is also sequentially weakly continuous in $(W^{1,p}(\Omega))^M$. In addition let f_3 satisfy the coercivity condition:

$$\exists (a, c) \in \left(\left(L^1(\Omega)\right)^M\right)^2, \exists n \in \mathbf{N}, \exists \beta_\nu > 1, d_\nu \geqslant b_\nu > 0 : \forall \mathbf{F} \in \mathbf{R}^{NM},$$
$$a(\mathbf{x}) + \sum_{\nu=1}^n b_\nu |\phi_\nu(\mathbf{F})|^{\beta_\nu} \leqslant f_3(\mathbf{x}, \mathbf{F}) \leqslant c(\mathbf{x}) + \sum_{\nu=1}^n d_\nu |\phi_\nu(\mathbf{F})|^{\beta_\nu}, \tag{2.9}$$

where ϕ_ν: $\mathbb{R}^{NM} \to \mathbb{R}$, $\nu = 1, 2, \ldots, n$, are null-Lagrangians (i.e., ϕ_ν and $-\phi_\nu$ are quasi-convex). Then, for every $\mathbf{u} \in (W^{1,\infty}(\Omega))^M$ such that $\mathbf{u} = \mathbf{u}_0$ on $\partial\Omega$ (in the sense of traces), $\mathbf{u}_0 \in (W^{1,\infty}(\Omega))^M$, there exist a sequence $\{\mathbf{u}^s\}_{s=1}^{\infty}$ such that

$$
\begin{aligned}
&\text{(i)} && \mathbf{u}^s \in \left(W^{1,\infty}(\Omega)\right)^M, \\
&\text{(ii)} && \forall s \geqslant 1, \mathbf{u}^s = \mathbf{u}_0 \text{ on } \partial\Omega \text{ (trace sense)}, \\
&\text{(iii)} && \phi_\nu(\nabla \mathbf{u}^s) \rightharpoonup \phi_\nu(\nabla \mathbf{u}) \text{ (weakly) in } \left(L^{\beta_\nu}(\Omega)\right)^M \text{ as } s \to \infty, \\
&\text{(iv)} && I(\mathbf{u}^s, \Omega) \to QI(\mathbf{u}, \Omega) \text{ as } s \to \infty. \qquad \square
\end{aligned}
\tag{2.10}
$$

In other words, this theorem establishes that:

$$\inf(2.8) = \inf(2.9). \tag{2.11}$$

2.3. *Lower and (Weak Lower) Semicontinuous Regularizations*

By definition the lower (weak lower) semicontinuous regularization $\tilde{F}$ of a given function is the greatest lower (weak lower) semicontinuous function everywhere less than F:

$$\tilde{F} = \sup_P \{P \leqslant F, P \text{ is lower (weak lower) semicontinuous}\}. \tag{2.12}$$

Equivalently, it can be shown that (see Ekeland and Temam [9, p. 10])

$$\forall \mathbf{u} \in V, \tilde{F}(\mathbf{u}) = \liminf_{\substack{\mathbf{v} \to \mathbf{u} \\ (\mathbf{v} \rightharpoonup \mathbf{u})}} F(\mathbf{v}), \tag{2.13}$$

$$\operatorname{epi} \tilde{F} = \overline{\operatorname{epi} F} \quad \text{(or weak closure)}, \tag{2.14}$$

where V is a locally convex space and where $F: V \to \overline{\mathbb{R}}$.

2.4. *Polyconvexity*

Basically, we say that a function of several variables is polyconvex if and only if it is convex in each of them. Specifically, using Ball's [2, p. 359] definition, we have the following:

Let $M^{n\times n}$ be the space of real $n \times n$ matrices with the induced norm of $\mathbb{R}^{n\times n}$ and let $U \subset M^{n\times n}$ be an open bounded subset of $M^{n\times n}$. Define the finite dimensional Euclidean spaces E and E_1 by:

$$E = E_1 \times \mathbb{R},$$

where

$$\begin{aligned} E_1 &= \varnothing && \Leftarrow n = 1, \\ E_1 &= M^{2\times 2} && \Leftarrow n = 2, \\ E_1 &= M^{3\times 3} \times M^{3\times 3} && \Leftarrow n = 3. \end{aligned}$$

That is, E may be identified with $\mathbb{R}^{s(n)}$, where $s(1) = 1$, $s(2) = 5$, $s(3) = 19$. Let us now define the map $T : M^{n\times n} \to E$ by:

$$\begin{aligned} T(\mathbf{F}) &= \mathbf{F} && \Leftarrow n = 1, \\ T(\mathbf{F}) &= (\mathbf{F}, \det \mathbf{F}) && \Leftarrow n = 2, \\ T(\mathbf{F}) &= (\mathbf{F}, \operatorname{adj} \mathbf{F}, \det \mathbf{F}) && \Leftarrow n = 3. \end{aligned}$$

Then a function $g: U \to \mathbb{R}$ is polyconvex if and only if

$$\exists G: T(U) \to \mathbb{R}: G \text{ is convex in } T(U),$$

where

$$\begin{aligned} g &= G && \Leftarrow n = 1, \\ g(\mathbf{F}) &= G(\mathbf{F}, \det \mathbf{F}) && \Leftarrow n = 2, \\ g(\mathbf{F}) &= G(\mathbf{F}, \operatorname{adj} \mathbf{F}, \det \mathbf{F}) && \Leftarrow n = 3. \end{aligned}$$

There are many interesting properties of polyconvex functions (Ball [2]), but for our purposes we are particularly interested in the following result, proved by Morrey [19] (see also Ball [2, p. 361]).

THEOREM 2.5. *If U is such that* $\operatorname{Co} T(U)$ *is open, then*

$$g \text{ is polyconvex} \Rightarrow g \text{ is quasi-convex on } U. \ \square$$

Remark 2.1. It can be also proved (Ball [2, 3], Morrey [18, 19]) that:

(a) Convexity $\Rightarrow$ Polyconvexity $\Rightarrow$ Quasi-convexity, (2.15)

(b) In the one-dimensional case:

Convexity $\Leftrightarrow$ Polyconvexity $\Leftrightarrow$ Quasi-convexity. (2.16)

2.5. *Polyconvexification*

Consider again the functional $I(\mathbf{u}, \Omega)$ as defined by (2.1) with $l = 1$. Then we define the *polyconvexification* Pf_1 of f_1 as:

$$Pf_1 = \sup_q \{q \leqslant f_1, q \text{ is polyconvex}\}. \tag{2.17}$$

We then define the functional $PI(\mathbf{u}, \Omega)(l = 1)$ as being:

$$PI(\mathbf{u}, \Omega) = \int_{\Omega} Pf_1(x, \nabla \mathbf{u})\, dx. \tag{2.18}$$

Remark 2.2. (a) From (2.6) and (2.15) we have for a positive integrand

$$Pf_1 \leqslant Qf_1 \Rightarrow PI(\mathbf{u}, \Omega) \leqslant QI(\mathbf{u}, \Omega). \tag{2.19}$$

2.6 *The Penalty Method*

Let Ω be an open bounded domain in $\mathbf{R}^N$ with a Lipschitzian boundary $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$ with $\partial\Omega_1 \cap \partial\Omega_2 = \varnothing$ and meas $\partial\Omega_1 > 0$.

Let K denote the set

$$K = \left\{\mathbf{u} \in \left(W^{1,p}(\Omega)\right)^N : \det(\mathbf{1} + \nabla \mathbf{u}) = 1 \text{ a.e. in } \Omega, \right.$$
$$\left. \mathbf{u} = \mathbf{o} \text{ a.e. in } \partial\Omega_1 \text{ (trace sense)}\right\} \tag{2.20}$$

We shall confine ourselves to cases for which $p \geqslant 2N$, $N = 1, 2$, or 3. This restriction on p can easily be relaxed later by using a continuity argument.

Consider the minimization problem,

$$\begin{cases} \text{Find } \mathbf{u} \in K, \text{ such that} \\ \pi(\mathbf{u}, \Omega) \leqslant \pi(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in K, \end{cases} \tag{2.21}$$

where π is the total potential energy functional

$$\pi(\mathbf{u}, \Omega) = \int_{\Omega} W(\mathbf{x}, \mathbf{1} + \nabla \mathbf{u})\, dx - f(\mathbf{u}). \tag{2.22}$$

Here W is the stored energy function per unit volume in the reference configuration, characterizing the mechanical response of the material, and is a positive function of the Carathéodory type (i.e., for almost all $\mathbf{x} \in \Omega$, $W(\mathbf{x}, \cdot)$ is continuous and for all $\nabla \mathbf{u}$ $W(\cdot, \mathbf{1} + \nabla \mathbf{u})$ is measurable on Ω). In (2.22), $-f(\mathbf{u})$ is the potential energy of the external forces and is typically of the form:

$$f(\mathbf{u}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{u}\, dx + \int_{\partial\Omega_2} \mathbf{t} \cdot \mathbf{u}\, ds, \qquad \forall \mathbf{u} \in \left(W^{1,p}(\Omega)\right)^N \tag{2.23}$$

with $\mathbf{f} \in ((W^{1,p}(\Omega))^N)'$ prescribed on Ω, representing the body force per unit volume of the reference configuration, and $\mathbf{t} \in ((W^{1-1/p,p}(\partial\Omega_2))^N)'$ prescribed on $\partial\Omega_2$ and representing the applied surface tractions per unit area of the reference configuration, $(1/p + 1/p' = 1)$.

The existence of solutions to (2.20) can be guaranteed by the generalized Weirstrass theorem. Consequently, there are two main issues concerning the existence of minimizers to (2.20), the sequentially weak closedness of the set of admissible motions K and the weak lower semicontinuity and coerciveness of the energy functional π. We now analyze these two cases.

We first show that K as defined by (2.20) is sequentially weakly closed. In order to do so we shall need the following results, the first one being a theorem by Ball [2, p. 369] and the second one a consequence of Ball [2, p. 372].

THEOREM 2.6. (i) *Let* $N = 2$; *then if* $\mathbf{w} \in (W^{1,2}(\Omega))^N$, $\det \nabla \mathbf{w} \in L^1(\Omega)$ *and*

$$\det \nabla \mathbf{w} = \tfrac{1}{2}\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} w,^i_\alpha w,^j_\beta w,^k_\gamma \tag{2.24}$$

(ii) Let $N = 3$; then if $w \in (W^{1,2}(\Omega))^N$, $\operatorname{adj} \nabla \mathbf{w} \in (L^1(\Omega))^{N^2}$ and

$$(\operatorname{adj} \nabla \mathbf{w})_i^\alpha = \tfrac{1}{2}\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} w,^j_\beta w,^k_\gamma = \left(\tfrac{1}{2}\varepsilon_{ijk}\varepsilon^{\alpha\beta\gamma} w,^j w^k_\gamma\right),_\beta \tag{2.25}$$

holds in $\mathfrak{D}'(\Omega)$.

(iii) Let $N = 3$; then if $w \in (W^{1,p}(\Omega))^N$ and $\operatorname{adj} \nabla \mathbf{w} \in (L^{p'}(\Omega))^{N^2}$, $\det \nabla \mathbf{w} \in L^1(\Omega)$, and (2.24) holds iN $\mathfrak{D}'(\Omega)$, $(1/p + 1/p' = 1, p \geqslant 2)$. □

THEOREM 2.7. (i) *Let* $N = 2$; *then if* $p > 2$ *the map* $\mathbf{w} \to \det \nabla \mathbf{w}$: $(W^{1,p}(\Omega))^N \to L^{p/2}(\Omega)$ *is sequentially weakly continuous.*

(ii) Let $N = 3$; then if $p > 2$ the map $\mathbf{w} \to \operatorname{adj} \nabla \mathbf{w} : (W^{1,p}(\Omega))^N \to (L^{p/2}(\Omega))^{N^2}$ is sequentially weakly continuous.

(iii) Let $N = 3$; then if $p > 3$ the map $\mathbf{w} \to \det \nabla \mathbf{w} : (W^{1,p}(\Omega))^N \to L^{p/3}(\Omega)$ is sequentially weakly continuous. □

PROPOSITION 2.8. *K as defined by* (2.20) *is sequentially weakly closed.*

Proof. Let $\{\mathbf{u}_n\} \in K$ be a sequence from K converging weakly to $\mathbf{u}$ in $(W^{1,p}(\Omega))^N$; then by definition of K and from Theorems 2.6 and 2.7 for all $\psi \in \mathfrak{D}(\Omega)$ we have

$$\int_\Omega [\det(\mathbf{1} + \nabla \mathbf{u}_n) - 1]\psi\, dx = 0 \overset{n,\infty}{\to} \int_\Omega [\det(\mathbf{1} + \nabla \mathbf{u}) - 1]\psi\, dx.$$

Consequently, $\det(\mathbf{1} + \nabla \mathbf{u}) = 1$ a.e. in Ω, where we assumed p and N to be given as in Theorems 2.6 and 2.7. From the linearity and continuity of the trace operator the result now follows. □

For the class of problems we wish to analyze here, W will not necessarily be quasi-convex. Thus, we wish to associate with this problem a relaxed one and establish a relationship between the original problem and the relaxed one. However, we must first develop a strategy to handle the constraint $\det(\mathbf{1} + \nabla \mathbf{u}) = 1$ a.e. in Ω. In order to do so, we shall use a Penalty Method. An advantage of using this method is of avoiding the problem of characterizing the correct space for the pressure term in the first Piola–Kirchhoff stress tensor.

Let us assume for the moment that π and K are such that the generalized Weirstrass theorem holds. Then the exterior penalty method for the constrained optimization problem (2.21) involves considering a penalty functional $P : A \to \mathbb{R}$ with the following properties:

(i) P is positive semidefinite on K in the sense that $P(\mathbf{v}) \geqslant 0$ for all $\mathbf{v} \in A$; $P(\mathbf{v}) = 0$ if $\mathbf{v} \in K$ and $P(\mathbf{v}) > 0$ if $\mathbf{v} \notin K$.

(ii) P is sequentially weakly lower semicontinuous.

Where

$$A = \left\{ \mathbf{v} \in \left(W^{1,p}(\Omega)\right)^n : \mathbf{v} = \mathbf{0} \qquad a.e.\ on\ \partial\Omega_1(\textit{trace sense}) \right\} \tag{2.26}$$

we then construct the penalized functional

$$\begin{cases} \pi_\varepsilon : A \to \overline{\mathbb{R}} \\ \pi_\varepsilon(\mathbf{v}, \Omega) = \pi(\mathbf{v}, \Omega) + \dfrac{1}{\varepsilon} P(\mathbf{v}) \end{cases} \tag{2.27}$$

where ε is a positive real number.

From the above conditions it is clear that π_ε is also proper, coercive and sequentially weakly lower semicontinuous on all of A. Thus there exists a solution $\mathbf{u}_\varepsilon \in A$ to the minimization problem

$$\begin{cases} \text{Find } \mathbf{u}_\varepsilon \in A, \text{ such that} \\ \pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) \leqslant \pi_\varepsilon(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in A \end{cases} \tag{2.28}$$

for every $\varepsilon > 0$.

The importance of the exterior penalty functional π_ε can be seen through the following result.

PROPOSITION 2.9. *Let π and K as defined above verify the conditions of the generalized Weirstrass theorem; then there exists at least one solution*

$\mathbf{u}_\varepsilon \in A$ *to the penalized optimization problem* (2.28). *Moreover, there exists a subsequence of solutions to* (2.28) *obtained as* ε *goes to zero which converges weakly in* A *to a solution* $\mathbf{u}$ *of the constrained optimization problem* (2.21).

Proof. The existence of $\mathbf{u}_\varepsilon$ solution of (2.28) was already established above.

We now observe that

$$\begin{cases} \pi_\varepsilon(\mathbf{v}, \Omega) \geqslant \pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) = \pi(\mathbf{u}_\varepsilon, \Omega) + \dfrac{1}{\varepsilon} P(\mathbf{u}_\varepsilon) \geqslant \pi(\mathbf{u}_\varepsilon, \Omega) \\ \forall \mathbf{v} \in A \end{cases}$$

Consequently

$$\pi(\mathbf{u}_\varepsilon, \Omega) \leqslant \pi(\mathbf{v}, \Omega) + \frac{1}{\varepsilon} P(\mathbf{v}), \qquad \forall \mathbf{v} \in A.$$

Choosing now $\mathbf{v} \in K$ we obtain

$$\pi(\mathbf{u}_\varepsilon, \Omega) \leqslant \pi(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in K. \tag{2.29}$$

Since π is coercive from (2.29), we see that there must exist a real constant $c > 0$ independent of ε such that $\|\mathbf{u}_\varepsilon\|_{1,p} \leqslant c$. Since A is a reflexive Banach space when endowed with the usual $\|\cdot\|_{1,p}$ norm, there exists a subsequence of solutions to (2.28) (also denoted by $\mathbf{u}_\varepsilon$) and an element $\mathbf{u} \in A$ such that $\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u}$ in A. Remarking now that for the above subsequence we have

$$\pi(\mathbf{u}_\varepsilon, \Omega) \leqslant \pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) \leqslant \inf_{\mathbf{v} \in K} \pi_\varepsilon(\mathbf{v}, \Omega) = \inf_{\mathbf{v} \in K} \pi(\mathbf{v}, \Omega).$$

Moreover, since π is sequentially weakly lower semicontinuous we must have

$$\liminf_{\varepsilon \to 0} \pi(\mathbf{u}_\varepsilon, \Omega) \geqslant \pi(\mathbf{u}, \Omega).$$

Consequently,

$$\pi(\mathbf{u}, \Omega) \leqslant \inf_{\mathbf{v} \in K} \pi(\mathbf{v}, \Omega).$$

In order to show that $\mathbf{u} \in K$ we note that there exists a $\mathbf{v}_0 \in K$, such that

$$\pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) = \pi(\mathbf{u}_\varepsilon, \Omega) + \frac{1}{\varepsilon} P(\mathbf{u}_\varepsilon) \leqslant \pi(\mathbf{v}_0, \Omega).$$

From the weak lower semicontinuity of P and the fact that π is coercive and that $\{\mathbf{u}_\varepsilon\}$ is bounded independently of ε, we have

$$0 \leqslant P(\mathbf{u}) \leqslant \liminf_{\varepsilon \to 0} P(\mathbf{u}_\varepsilon) \leqslant \lim_{\varepsilon \to 0} \left[\pi(\mathbf{v}_0, \Omega) - \pi(\mathbf{u}_\varepsilon, \Omega)\right]\varepsilon = 0$$

That is,

$$P(\mathbf{u}) = 0 \Rightarrow \mathbf{u} \in K$$

and therefore $\mathbf{u}$ is a solution to (2.21). □

The advantages of considering an exterior penalty formulation to the optimization problem (2.21) reside on the fact that the minimizers of π_ε can be sought in the entire space A because the constraint set K enters the problem only in the construction of the penalty functional P, and also on the fact that minimizers $\mathbf{u}_\varepsilon$ of π_ε can be chosen so as to approximate an actual minimizer of π arbitrarily closely in the weak topology of A by taking ε sufficiently small.

Among all the possible penalty functionals we shall use

$$\begin{cases} P(\mathbf{v}) = \frac{1}{2}\int_\Omega [\det(\mathbf{1} + \nabla \mathbf{v}) - 1]^2 \, dx, \\ \forall \mathbf{v} \in A \end{cases} \tag{2.30}$$

which as we shall see has also the advantage of giving an explicit way of calculating the hydrostatic part of the stress tensor. In the above the exponent p is assumed to be such that $\det(\mathbf{1} + \nabla \mathbf{v}) \in L^2(\Omega)$.

From its definition it is clear that (2.30) verifies condition (i) of the definition of a penalty term. The sequential weak lower semicontinuity of (2.30) is an immediate consequence of Theorem 2.7 and of the following result for a proof of which we refer to Oden and Kikuchi [23, p. 10].

THEOREM 2.10. *Let $\mathcal{U}$ and $\mathcal{V}$ be normed linear spaces and let K be a (sequentially) weakly closed subset of $\mathcal{U}$. Let π: $K \to \mathbb{R}$ be a functional defined by*

$$\pi = GoB; \qquad \pi(\mathbf{u}) = G(B(\mathbf{u})), \qquad \forall \mathbf{u} \in K,$$

where

B: $K \to \mathcal{V}$ is a (sequentially) weakly continuous map

G: $M \to \mathbb{R}$ is a convex Gâteaux differentiable functional defined on a convex set $M \subset \mathcal{V}$ containing the range of B ($Rg(B)$).

Then π is (sequentially) weakly lower semicontinuous on K. □

2.7. *Relaxation.*

With these definitions in mind we observe that $\pi(\pi_\varepsilon)$ is sequential weak lower semicontinuous whenever $W(\cdot,\cdot)$ $(W(\cdot,\cdot) + (1/2\varepsilon)[\det(\cdot) - 1]^2)$ is quasi-convex (on $\nabla\mathbf{v}$) and of class $\mathcal{T}_p(\Omega)$, by Theorem 2.1.; or when

$$W(\mathbf{x}, \mathbf{1} + \nabla\mathbf{v}) = \psi_1(\mathbf{x}, \mathbf{1} + \nabla\mathbf{v}) + \psi_2(\mathbf{x}, \operatorname{adj}(\mathbf{1} + \nabla\mathbf{v})) \tag{2.31}$$

for all the values of the arguments, where

$$\begin{aligned} \psi_1 &: \Omega \times \left(L^p(\Omega)\right)^{n^2} \to L^1(\Omega) \\ \psi_2 &: \Omega \times \left(L^{p/2}(\Omega)\right)^{n^2} \to L^1(\Omega) \end{aligned} \tag{2.32}$$

in which $\psi_1(\cdot,\cdot)$ and $\psi_2(\cdot,\cdot)$ are positive Carathéodory, Gâteaux differentiable, convex functions in $\mathbf{1} + \nabla\mathbf{v}$ and $\operatorname{adj}(\mathbf{1} + \nabla\mathbf{v})$, respectively, due to Theorem 2.10.

In order to prove, by using the generalized Weirstrass theorem, that the infinum is actually attained it only remains to prove coercivity.

PROPOSITION 2.11. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitzian boundary $\partial\Omega = \overline{\partial\Omega_1} \cup \overline{\partial\Omega_2}$, $\partial\Omega_1 \cap \partial\Omega_2 = \phi$ and with meas $\partial\Omega_1 > 0$. Let there exist $a(\mathbf{x}) \in L^1(\Omega)$, and real numbers $k_1 > 0$, $k_2 \geqslant 2$ such that for all $\mathbf{v} \in A$ and almost every $\mathbf{x} \in \Omega$, the following holds*:

$$a(\mathbf{x}) + k_1|\mathbf{1} + \nabla\mathbf{v}|^p + k_2|\operatorname{adj}(\mathbf{1} + \nabla\mathbf{v})|^q \leqslant QW(\mathbf{x}, \mathbf{1} + \nabla\mathbf{v}) \tag{2.33}$$

where $q \geqslant p/2$. Then $Q\pi$ is coercive on $(W^{1,p}(\Omega))^n$.

Proof. From the trace theorems (Adams [1, p. 216]) $\mathbf{v}|_{\partial\Omega} \in (W^{1-1/p,p}(\partial\Omega))^n$ and as $(W^{1-1/p,p}(\partial\Omega))^n \subset (L^p(\partial\Omega))^n$, we have

$$\exists c_1 \in \mathbb{R}^+ \setminus \{0\} : F(\mathbf{v}) \leqslant \left(\|\mathbf{f}\|_{(1,p)'} + c_1\|\mathbf{t}\|_{(1-1/p,p)',\partial\Omega_2}\right)\|\mathbf{v}\|_{1,p,\Omega}.$$

From the hypothesis on QW we obtain for all $\mathbf{v}$ belonging to $(W^{1,p}(\Omega))^n$

$$\begin{aligned} Q\pi(\mathbf{v}, \Omega) \geqslant{}& \int_\Omega a(\mathbf{x})\, dx + k_1\|\mathbf{1} + \nabla\mathbf{v}\|^p_{0,p} + k_2\|\operatorname{adj}(\mathbf{1} + \nabla\mathbf{v})\|^q_{0,q} \\ &- \left(\|\mathbf{f}\|_{(1,p)'} + c_1\|\mathbf{t}\|_{(1-1/p,p)',\partial\Omega_2}\right)\|\mathbf{v}\|_{1,p}. \end{aligned}$$

Since $\partial\Omega_1$ has positive measure a result of Morrey [19, p. 82] implies that there exists a real number $c_2 > 0$, such that

$$\|\mathbf{I} + \mathbf{v}\|^p_{0,p} \leqslant c_2\left[\|\mathbf{1} + \nabla\mathbf{v}\|^p_{0,p} + \left(\int_{\partial\Omega_1} |\mathbf{x}|\, ds\right)^p\right]$$

where **I** represents the identity function, and we obtain

$$\begin{aligned} Q\pi(\mathbf{v}, \Omega) \geqslant & \int_\Omega a(\mathbf{x})\, dx + \frac{k_1}{2}\|\mathbf{1} + \nabla \mathbf{v}\|_{0,p}^p \\ & + \frac{k_1}{2}\left[\frac{1}{c_2}\|\mathbf{I} + \mathbf{v}\|_{0,p}^p - \left(\int_{\partial\Omega_1} |\mathbf{x}|\, ds\right)^p\right] \\ & + k_2\|\operatorname{adj}(\mathbf{1} + \nabla \mathbf{v})\|_{0,q}^q - \left(\|\mathbf{f}\|_{(1,p)'} + c_1\|\mathbf{t}\|_{(1-1/p,p)',\partial\Omega_2}\right)\|\mathbf{v}\|_{1,p}. \end{aligned}$$

Young's inequality asserts that

$$\forall\, (a, b, \delta) \in (\mathbb{R}^+ \setminus \{0\})^3$$

$$ab \leqslant \delta a^p + \frac{1}{p'(\delta p)^{p'/p}} b^{p'}; \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

Thus,

$$\begin{aligned} & \left(\|\mathbf{f}\|_{(1,p)'} + c_1\|\mathbf{t}\|_{(1-1/p,p)',\partial\Omega_2}\right)\|\mathbf{v}\|_{1,p} \\ & \qquad \leqslant \delta\|\mathbf{v}\|_{1,p}^p + \frac{1}{p'(\delta p)^{p'/p}}\left(\|\mathbf{f}\|_{(1,p)'} + c_1\|\mathbf{t}\|_{(1-1/p,p)',\partial\Omega_2}\right)^{p'}. \end{aligned}$$

Consequently,

$$\begin{aligned} Q\pi(\mathbf{v}, \Omega) \geqslant & \int_\Omega a(\mathbf{x})\, dx + \left[\min\left(\frac{k_1}{2}, \frac{k_1}{2c_2}\right)\right]\|\mathbf{I} + \mathbf{v}\|_{1,p}^p - \delta\|\mathbf{v}\|_{1,p}^p \\ & + k_2\|\operatorname{adj}(\mathbf{1} + \nabla \mathbf{v})\|_{0,q}^q - \frac{k_1}{2}\left(\int_{\partial\Omega_1} |\mathbf{x}|\, ds\right)^p \\ & - \frac{1}{p'(\delta p)^{p'/p}}\left(\|\mathbf{f}\|_{(1,p)'} + c_1\|\mathbf{t}\|_{(1-1/p,p)',\partial\Omega_2}\right)^{p'}. \end{aligned} \tag{2.34}$$

Choosing now $0 < \delta < \min(k_1/2, k_1/2c_2)$ and using the fact that

$$\|\mathbf{v}\|_{1,p}^p \leqslant \left(\|\mathbf{I} + \mathbf{v}\|_{1,p} + \|\mathbf{I}\|_{1,p}\right)^p$$

the result follows. □

Remark 2.12. (i) If in (2.31) we have PW or W then an analogous conclusion can be drawn for those cases.

(ii) If the result holds for $P\pi$ then it holds for $Q\pi$, π.

(iii) If the result holds for $P\pi_\varepsilon$ then it holds for $Q\pi_\varepsilon$ and π_ε.

We shall now summarize all the problems we wish to consider, give sufficient conditions for the existence of solutions, and then relate these solutions.

The basic problem under consideration is (2.21)

$$\begin{cases} \text{Find } \mathbf{u} \in K, \text{ such that} \\ \pi(\mathbf{u}, \Omega) \leqslant \pi(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in K \end{cases} \tag{2.21}$$

where π and K are defined in (2.20), (2.22), and (2.23).

We associate with it the following problems

$$\begin{cases} \text{Find } \mathbf{u} \in K, \text{ such that} \\ Q\pi(\mathbf{u}, \Omega) \leqslant Q\pi(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in K. \end{cases} \tag{2.35}$$

Its polyconvexification consists of

$$\begin{cases} \text{Find } \mathbf{u} \in K, \text{ such that} \\ P\pi(\mathbf{u}, \Omega) \leqslant P\pi(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in K. \end{cases} \tag{2.36}$$

The corresponding penalized versions are, respectively,

$$\begin{cases} \text{Find } \mathbf{u}_\varepsilon \in A, \text{ such that} \\ \pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) \leqslant \pi_\varepsilon(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in A \end{cases} \tag{2.28}$$

where π_ε is given by (2.27) with (2.30).

$$\begin{cases} \text{Find } \mathbf{u}_\varepsilon \in A, \text{ such that} \\ Q\pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) \leqslant Q\pi_\varepsilon(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in A \end{cases} \tag{2.37}$$

and

$$\begin{cases} \text{Find } \mathbf{u}_\varepsilon \in A, \text{ such that} \\ P\pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) \leqslant P\pi(\mathbf{v}, \Omega), \qquad \forall \mathbf{v} \in A. \end{cases} \tag{2.38}$$

The following results are then a consequence of the previous study and of the generalized Weirstrass Theorem.

PROPOSITION 2.13. *Let K be defined as in* (2.20); *then the following holds*:

(i) If π is sequentially weakly lower semicontinuous and if $W(\cdot, \cdot)$ verifies (2.33) (instead of QW) then the infinum of (2.21) is attained on K.

(ii) If $P\pi$ is of the form (2.31)–(2.32) and if it verifies (2.33) (instead of QW) then the infinum of (2.36) is attained on K.

(iii) If $Q\pi$ is of class $\mathfrak{T}_p$ or of the form (2.31)–(2.32) and if it verifies (2.33) then the infinum of (2.35) is attained on K. □

PROPOSITION 2.14. *A result analogous to the previous one can be obtained for each* $\varepsilon > 0$ *if we replace* K *by* A *and* π, $P\pi$, $Q\pi$, *by* π_ε, $P\pi_\varepsilon$, *and* $Q\pi_\varepsilon$, *respectively, in Proposition* 2.13. □

The next result relates the solutions of (2.28) and of (2.37) and is a consequence of Proposition 2.2.

PROPOSITION 2.15. *Let* $W(\cdot,\cdot)$ *as defined in* (2.22) *be a positive Carathéodory function. Let* $f(\cdot,\cdot)$ *be as defined by* (2.23) *with* $\partial\Omega_2 = \phi$, *and identify it with* $f_4(\cdot,\cdot)$ *of Proposition* 2.2. *Let* $W(\mathbf{x}, \mathbf{1} + \nabla\mathbf{v}) + (1/2\varepsilon)[\det(\mathbf{1} + \nabla\mathbf{v}) - 1]^2$ *verify* (2.9) *and identify it with* $f_3(\mathbf{x}, \nabla\mathbf{v})$ *of Proposition* 2.2 *for all* $\mathbf{x} \in \Omega$ *and all* $\mathbf{v} \in (W^{1,p}(\Omega))^n$. *Then for every* $\varepsilon > 0$ *and every* $\mathbf{u}_\varepsilon \in (W^{1,\infty}(\Omega))^n$ *such that* $\mathbf{u}_\varepsilon = \mathbf{u}_0$ *on* $\partial\Omega$ *(in the trace sense),* $\mathbf{u}_0 \in (W^{1,\infty}(\Omega))^n$, *there exists a sequence* $\{\mathbf{u}_\varepsilon^\sigma\}_{\sigma=1}^\infty$ *such that*

(i) $\forall \sigma \geqslant 1,\ \mathbf{u}_\varepsilon^\sigma \in (W^{1,\infty}(\Omega))^n$.

(ii) $\forall \sigma \geqslant 1,\ \mathbf{u}_\varepsilon^\sigma = u_0$ on $\partial\Omega$ (trace sense).

(iii) $\forall \nu = 1, 2, \ldots, n_0\ \phi_\nu(\nabla \mathbf{u}_\varepsilon^\sigma) \rightharpoonup \phi_\nu(\nabla \mathbf{u}_\varepsilon)$ *weakly in* $L^{\beta_\nu}(\Omega)$ *as* $\sigma \to \infty$.

(iv) *If* (iii) *implies that* $\mathbf{u}_\varepsilon^\sigma \rightharpoonup \mathbf{u}_\varepsilon$ *in* $(W^{1,p}(\Omega))^n$ *then* $\pi_\varepsilon(\mathbf{u}_\varepsilon^\sigma, \Omega) \to Q\pi_\varepsilon(\mathbf{u}, \Omega)$ *as* $\sigma \to \infty$.

In other words, $\inf \pi_\varepsilon = \inf Q\pi_\varepsilon$ over all $\mathbf{v} \in (W^{1,p}(\Omega))^n$ such that $\mathbf{v} = \mathbf{u}_0$ on $\partial\Omega$.

Remark 2.16. (i) If in addition to Proposition 2.15 π_ε and $Q\pi_\varepsilon$ are as in Propositions 2.13 and 2.14, then the above infinum is actually attained.

(ii) If in addition $Q\pi_\varepsilon(\mathbf{v}, \Omega) = Q\pi(\mathbf{v}, \Omega) + \int_\Omega (1/2\varepsilon)[\det(\mathbf{1} + \nabla\mathbf{v}) - 1]^2\, dx$ for all $\mathbf{v} \in (W^{1,p}(\Omega))^n$ then a conclusion analogous to the one in Proposition 2.9 can be obtained.

(iii) If in addition $P\pi_\varepsilon(\mathbf{v}, \Omega) \equiv Q\pi_\varepsilon(\mathbf{v}, \Omega)$ and if $P\pi_\varepsilon$ is as in Propositions 2.13 and 2.14 then the infinum in Proposition 2.15 is also attained for $P\pi_\varepsilon$. Moreover if $P\pi_\varepsilon(\mathbf{v}, \Omega) = P\pi(\mathbf{v}, \Omega) + \int_\Omega (1/2\varepsilon)[\det(\mathbf{1} + \nabla\mathbf{v}) - 1]^2\, dx$ (for example, in the incompressible case) then a conclusion analogous to (ii) also holds.

(iv) If $Q\pi_\varepsilon$ in Proposition 2.15 is of class $\mathfrak{T}_p$ or of the form (2.31)–(2.32), i.e., sequentially weakly lower semicontinuous, then the infinum in Proposition 2.15 is also equal to $\inf \tilde{\pi}_\varepsilon(\mathbf{v}, \Omega)$ for all $\mathbf{v}$ belonging to $(W^{1,p}(\Omega))^n$ such that $\mathbf{v} = \mathbf{u}_0$ a.e. on $\partial\Omega$ (trace sense).

We now wish to relate the solutions of (2.21), (2.28), (2.35)–(2.38) (when they exist). In order to do so we shall need some preliminary results, which follow Ekeland [10].

PROPOSITION 2.17. *Let* K *be a (sequentially) weakly closed set of a complete metric space* V. *Let* $F : K \to \overline{\mathbf{R}}$ *be a bounded from below* ($\inf F >$

$-\infty$) *lower-semicontinuous functional on* K. *Let* $\lambda > 0$ *and* $\mathbf{u} \in K$ *be given such that*:

$$F(\mathbf{u}) \leqslant \inf_{\mathbf{v} \in K} F(\mathbf{v}) + \lambda.$$

Then there exists a $\mathbf{u}_\lambda \in K$ *which is a better minimizer of* F *than* $\mathbf{u}$ *and which is the vertex of a cone entirely below graph* F. *In particular*,

(a) $F(\mathbf{u}_\lambda) \leqslant F(\mathbf{u}) \leqslant \inf_{\mathbf{v} \in K} F(\mathbf{v}) + \lambda$,

(b) $d(\mathbf{u}, \mathbf{u}_\lambda) \leqslant 1$,

(c) $\forall \mathbf{v} \in K: \mathbf{v} \neq \mathbf{u}_\lambda, \qquad F(\mathbf{v}) > F(\mathbf{u}_\lambda) - \lambda d(\mathbf{v}, \mathbf{u}_\lambda)$.

Proof. For any $\alpha > 0$ we introduce an ordering on $K \times \mathbf{R}$ in the following way:

$$(\mathbf{v}_2, a_2) \succcurlyeq (\mathbf{v}_1, a_1) \overset{\text{iff}}{\Leftrightarrow} (a_2 - a_1) + \alpha d(\mathbf{v}_1, \mathbf{v}_2) \leqslant 0$$

and claim that for any $(\mathbf{u}_1, a_1) \in \operatorname{epi} F$, there exists $(\bar{\mathbf{u}}, \bar{a}) \in \operatorname{epi} F$ which is maximal in epi F with respect to the above ordering. In fact, let us define inductively a sequence $\mathbf{u}_n$, $n \in \mathbf{N}$ and let

$$S_n = \{(\mathbf{u}, a) \in \operatorname{epi} F: (\mathbf{u}, a) \succcurlyeq (\mathbf{u}_n, a_n)\}$$

and

$$m_n = \inf\{a: (\mathbf{u}, a) \in S_n\}.$$

Then pick any $(\mathbf{u}_{n+1}, a_{n+1}) \in S_n$ such that

$$(a_n - a_{n+1}) \geqslant \tfrac{1}{2}(a_n - m_n),$$

which implies that

$$(a_{n+1} - m_{n+1}) \leqslant (a_{n+1} - m_n) \leqslant \tfrac{1}{2}(a_n - m_n)$$
$$\leqslant \frac{1}{2^n}(a_1 - m_1) = \frac{1}{2^n}(a_1 - \inf F)$$

and as $\inf F > -\infty$, $a_{n+1} - m_{n+1} \to 0$ as $n \to \infty$, that is to say, the height of the one containing S_n goes to zero as n goes to ∞.

On the other hand, let $(\mathbf{u}, a) \in S_n$ be an arbitrary element of S_n. Then

$$(\mathbf{u}, a) \succcurlyeq (\mathbf{u}_n, a_n) \Leftrightarrow (a - a_n) + \alpha d(\mathbf{u}, \mathbf{u}_n) \leqslant 0$$
$$\Leftrightarrow \alpha d(\mathbf{u}, \mathbf{u}_n) \leqslant (a_n - a)$$
$$\Rightarrow d(\mathbf{u}, \mathbf{u}_n) \leqslant \frac{1}{\alpha}(a_n - m_n) \leqslant \frac{1}{\alpha}\frac{1}{2^n}(a_1 - m_1),$$

which goes to zero as n goes to ∞. We have thus obtained a sequence of nested sets (by definition) whose diameter goes to zero. Also from the above we conclude that $\mathbf{u}_n$ is a Cauchy sequence on V and as V is complete it is convergent on V. Therefore, $\exists \bar{\mathbf{u}} \in V$ s.t. $\mathbf{u}_n \to \bar{\mathbf{u}}$. But $\bar{\mathbf{u}} \in K$ because $\mathbf{u}_n \in K$ and K is (sequentially) weakly closed and therefore strongly closed. Moreover, being F lower semicontinuous its epigraph is closed and we have:

$$\exists \bar{a} \in \mathbf{R} \text{ s.t. } \{(\mathbf{u}, a)\} = \bigcap_{n=1}^{\infty} S_n; \qquad (\bar{\mathbf{u}}, \bar{a}) \in \text{epi } F \text{ is maximal.}$$

In fact, if by contradiction $\exists(\tilde{\mathbf{u}}, \tilde{a}) \in \text{epi } F$ such that $(\tilde{\mathbf{u}}, \tilde{a}) \succcurlyeq (\bar{\mathbf{u}}, \bar{a})$

$$\begin{aligned} &(\tilde{\mathbf{u}}, \tilde{a}) \succcurlyeq (\bar{\mathbf{u}}, \bar{a}) \succcurlyeq (\mathbf{u}_n, a_n), \qquad \forall n \\ &\Rightarrow (\tilde{\mathbf{u}}, \tilde{a}) \in S_n, \forall n \Rightarrow (\tilde{\mathbf{u}}, \tilde{a}) \bigcap_n S_n \\ &\Rightarrow (\tilde{\mathbf{u}}, \tilde{a}) = (\bar{\mathbf{u}}, \bar{a}) \Rightarrow (\bar{\mathbf{u}}, \bar{a}) \text{ is maximal.} \end{aligned}$$

Using now this property and identifying $(\mathbf{u}_1, a_1)$ with $(\mathbf{u}, F(\mathbf{u}))$ we conclude that

$$\exists(\mathbf{u}_\lambda, a_\lambda) \succcurlyeq (\mathbf{u}, F(\mathbf{u})) \text{ and maximal in epi } F.$$

In fact:

$$(\mathbf{u}_\lambda, a_\lambda) = (\mathbf{u}_\lambda, F(\mathbf{u}_\lambda))$$

because if by contradiction $a_\lambda > F(\mathbf{u}_\lambda) \Rightarrow (\mathbf{u}_\lambda, F(\mathbf{u}_\lambda)) \succcurlyeq (\mathbf{u}_\lambda, a_\lambda) \Rightarrow (\mathbf{u}_\lambda, a_\lambda)$ is not maximal. Also, by definition $F(\mathbf{u}_\lambda) \leqslant F(\mathbf{u})$ and $(\mathbf{u}_\lambda, F(\mathbf{u}_\lambda)) \succcurlyeq (\mathbf{u}, F(\mathbf{u}))$, which implies that

$$\begin{aligned} (F(\mathbf{u}_\lambda) - F(\mathbf{u})) + \lambda d(\mathbf{u}, \mathbf{u}_\lambda) &\leqslant 0 \\ \Leftrightarrow \lambda d(\mathbf{u}, \mathbf{u}_\lambda) &\leqslant F(\mathbf{u}) - F(\mathbf{u}_\lambda) \\ &\leqslant F(\mathbf{u}) - \inf_{\mathbf{v} \in K} F(\mathbf{v}) \\ &\leqslant \lambda \\ \Rightarrow d(\mathbf{u}, \mathbf{u}_\lambda) &\leqslant 1 \end{aligned}$$

and finally assuming by contradiction that $\forall \mathbf{v} \in K$ such that $v \neq \mathbf{u}_\lambda$ that $F(\mathbf{v}) > F(\mathbf{u}_\lambda) - \lambda d(\mathbf{v}, \mathbf{u}_\lambda)$ does not hold, we have:

$$\begin{aligned} &\exists \mathbf{v} \in K: \mathbf{v} \neq \mathbf{u}_\lambda \quad \text{and} \quad F(\mathbf{v}) \leqslant F(\mathbf{u}_\lambda) - \lambda d(\mathbf{v}, \mathbf{u}_\lambda) \\ &\Rightarrow F(\mathbf{v}) - F(\mathbf{u}_\lambda) + \lambda d(\mathbf{v}, \mathbf{u}_\lambda) \leqslant 0 \\ &\Rightarrow \text{epi } F \ni (\mathbf{v}, F(\mathbf{v})) \succcurlyeq (\mathbf{u}_\lambda, F\mathbf{u}_\lambda) \\ &\Rightarrow (\mathbf{u}_\lambda, F\mathbf{u}_\lambda) \text{ is not maximal, a contradiction. } \square \end{aligned}$$

PROPOSITION 2.18. *Let K and A be defined as in* (2.20) *and* (2.26), *respectively, with $\partial\Omega_2 = \phi$ and $\mathbf{v} = \mathbf{u}_0$* a.e. *on $\partial\Omega$ (trace sense).*

Let π, π_ε, $Q\pi$, $Q\pi_\varepsilon$, $P\pi$, $P\pi_\varepsilon$ be defined as in (2.22), (2.23), (2.27), (2.30); (2.6), (2.7), *and* (2.17), (2.18), *respectively. Let the hypothesis of Proposition* 2.15 *and of Remark* 2.16 *hold; then if $P\pi(P\pi_\varepsilon)$ verifies* (ii) *of Proposition* 2.13 (2.14), *we have*

$$\begin{aligned}\inf_{\mathbf{v}\in K} \pi(\mathbf{v},\Omega) &= \inf_{\mathbf{v}\in K} Q\pi(\mathbf{v},\Omega) = \min_{\mathbf{v}\in K} P\pi(\mathbf{v},\Omega) = \inf_{\mathbf{v}\in K} \tilde{\pi}(\mathbf{v},\Omega)\\ &= \inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v},\Omega) = \inf_{\mathbf{v}\in A} Q\pi_\varepsilon(\mathbf{v},\Omega) = \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v},\Omega)\\ &= \inf_{\mathbf{v}\in A} \tilde{\pi}_\varepsilon(\mathbf{v},\Omega).\end{aligned}$$

Proof. By definition of weak lower-semicontinuous regularization, quasi-convexification, polyconvexification and by Propositions 3.2 and 3.3, we have:

$$\begin{aligned}\inf_{\mathbf{v}\in K} \pi(\mathbf{v},\Omega) &\geqslant \inf_{\mathbf{v}\in K} Q\pi(\mathbf{v},\Omega) \geqslant \min_{\mathbf{v}\in K} P\pi(\mathbf{v},\Omega) \geqslant \inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v},\Omega)\\ &= \inf_{\mathbf{v}\in A} Q\pi_\varepsilon(\mathbf{v},\Omega) = \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v},\Omega) = \inf_{\mathbf{v}\in A} \tilde{\pi}_\varepsilon(\mathbf{v},\Omega)\end{aligned}$$

and also

$$\inf_{\mathbf{v}\in K} \pi(\mathbf{v},\Omega) \geqslant \inf_{\mathbf{v}\in K} \tilde{\pi}(\mathbf{v},\Omega) \geqslant \min_{\mathbf{v}\in K} P\pi(\mathbf{v},\Omega).$$

By Proposition 3.6 we have $\forall\varepsilon > 0\ \exists\mathbf{u}_\varepsilon \in A$ such that

$$P\pi_\varepsilon(\mathbf{u}_\varepsilon,\Omega) = \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v},\Omega).$$

and there exists $\mathbf{u}_{\varepsilon'} \rightharpoonup \bar{\mathbf{u}} \in K$ such that:

$$\begin{aligned}\min_{\mathbf{v}\in K} P\pi(\mathbf{v},\Omega) &= P\pi(\bar{\mathbf{u}},\Omega) \leqslant P\pi(\mathbf{u}_{\varepsilon'},\Omega) \leqslant P\pi_{\varepsilon'}(\mathbf{u}_{\varepsilon'},\Omega)\\ &\leqslant P\pi_\varepsilon(\mathbf{v},\Omega), \forall\mathbf{v} \in A \Rightarrow P\pi(\bar{\mathbf{u}}) \leqslant \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v}).\end{aligned}$$

Consequently,

$$P\pi(\bar{\mathbf{u}},\Omega) = P\pi_\varepsilon(\bar{\mathbf{u}},\Omega) = \min_{\mathbf{v}\in K} P\pi(\mathbf{v},\Omega) = \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v},\Omega) = \min_{\mathbf{v}\in K} P\pi_\varepsilon(\mathbf{v},\Omega).$$

That is, $P\pi_\varepsilon$ attains its minimum on K.
Let now $\delta > 0$ and $\mathbf{u} \in K$ be such that:

$$P\pi_\varepsilon(\mathbf{u},\Omega) \leqslant \pi_\varepsilon(\mathbf{u},\Omega) \leqslant \inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v},\Omega) + \delta = \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v},\Omega) + \delta.$$

Using now the fact that $P\pi_\varepsilon$ is sequentially weakly lower semicontinuous (by Theorem 2.10) we obtain, by Proposition 2.17, that $\exists \mathbf{u}^{\delta_1} \in K$, which is a "better" minimizer of $P\pi_\varepsilon$ on A in the sense that

$$P\pi(u^{\delta_1}, \Omega) = P\pi_\varepsilon(\mathbf{u}^{\delta_1}, \Omega) \leqslant P\pi_\varepsilon(\mathbf{u}, \Omega) \leqslant \pi_\varepsilon(\mathbf{u}, \Omega) \leqslant \inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v}, \Omega) + \delta_1.$$

Choosing now $\delta_1 > \delta_2 > \cdots \delta_n > \cdots > 0$, we obtain a bounded minimizing sequence $\{\mathbf{u}^{\delta_n}\}_{n=1}^{\infty}$ on K (due to (2.33)) and, consequently, there exists a weakly convergent subsequence $\{u^{\delta_{n'}}\}$ to $\hat{\mathbf{u}} \in K$ (because K is weakly sequentially closed). From the weak lower semicontinuity of $P\pi_\varepsilon$ we have:

$$\min_{\mathbf{v}\in K} P\pi_\varepsilon(\mathbf{v}, \Omega) = P\pi_\varepsilon(\hat{\mathbf{u}}, \Omega) \leqslant P\pi_\varepsilon(\mathbf{u}^{\delta_{n'}}, \Omega)$$

but $\min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v}, \Omega) = \inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v}, \Omega) = \inf_{\mathbf{v}\in A} \tilde{\pi}_\varepsilon(\mathbf{v}, \Omega) \Rightarrow \mathbf{u}^{\delta_{n'}} \in K$ is a minimizing sequence of π_ε on A and obviously of π on K. Consequently,

$$\inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v}, \Omega) = \inf_{\mathbf{v}\in K} \pi(\mathbf{v}, \Omega). \ \square$$

3. Generalized Solutions

We now establish in what sense solutions to $P\pi(P\pi_\varepsilon)$ can be seen as generalized solutions to $\pi(\pi_\varepsilon)$.

Proposition 3.1. (i) *Let* $\mathbf{u}_\varepsilon$ *be a solution of* $\min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v}, \Omega)$. *Then* $\exists \mathbf{u}^n$: $\mathbf{u}_\varepsilon^n \rightharpoonup \mathbf{u}_\varepsilon$ *in* A *and* $\mathbf{u}_\varepsilon^n$ *is a minimizing sequence of* $\pi_\varepsilon(\mathbf{v}, \Omega)$ *on* A.

(ii) *Let* $\mathbf{u}_\varepsilon^n$ *be a minimizing sequence of* $\pi_\varepsilon(\mathbf{v}, \Omega)$ *on* A. *Then* $\exists \mathbf{u}_\varepsilon^n k$ *(a subsequence)*: $\mathbf{u}_\varepsilon^n k \rightharpoonup \mathbf{u}_\varepsilon$ *on* A *and* $\mathbf{u}_\varepsilon$ *is a solution of* $\min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v}, \Omega)$.

Proof. (i) Let $\mathbf{u}_\varepsilon$ be a solution of $\min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v}, \Omega)$. Then by Proposition 2.18

$$P\pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) = \liminf_{u_\varepsilon^n \rightharpoonup u_\varepsilon} \pi_\varepsilon(\mathbf{u}_\varepsilon^n, \Omega).$$

But

$$P\pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega) = \min_{\mathbf{v}\in A} P\pi_\varepsilon(\mathbf{v}, \Omega) = \inf_{\mathbf{v}\in A} \pi_\varepsilon(\mathbf{v}, \Omega).$$

Therefore $\mathbf{u}_\varepsilon^n$ is a minimizing sequence of $\pi_\varepsilon(\mathbf{v}, \Omega)$ on A.

(ii) Let now $\mathbf{u}_\varepsilon^n$ be a minimizing sequence of $\pi_\varepsilon(\mathbf{v}, \Omega)$ on A. From the proof of Proposition 3.8, $\pi_\varepsilon \geqslant P\pi_\varepsilon$, which is bounded below. Then $\mathbf{u}_\varepsilon^n$ is bounded, which implies

$$\exists \mathbf{u}_\varepsilon^{n_k}: \mathbf{u}_\varepsilon^{n_k} \rightharpoonup \mathbf{u}_\varepsilon \qquad \text{on } A$$

but u_ε^n is a minimizing sequence. Therefore

$$\liminf_{\mathbf{u}_\varepsilon^{n_k} \rightharpoonup \mathbf{u}_\varepsilon} \pi_\varepsilon(\mathbf{u}_\varepsilon^{n_k}, \Omega) = \inf_{\mathbf{v} \in A} P\pi_\varepsilon(\mathbf{v}, \Omega) = P\pi_\varepsilon(\mathbf{u}_\varepsilon, \Omega). \ \square$$

PROPOSITION 3.2. (i) *Let* $\mathbf{u}$ *be a solution of* $\min_{\mathbf{v} \in K} P\pi(\mathbf{v}, \Omega)$. *Then*:

$$\exists \mathbf{u}^n: \mathbf{u}^n \rightharpoonup \mathbf{u} \qquad \text{on } K$$

and $\mathbf{u}^n$ *is a minimizing sequence of* $\pi(\mathbf{v}, \Omega)$ *on* K.

(ii) *There exists a subsequence* $\mathbf{u}^{n_k}$ *of the minimizing sequence* $\mathbf{u}^n$ *of* (i) *and a function* $\mathbf{u} \in K$ *such that* $\mathbf{u}^{n_k} \rightharpoonup \mathbf{u}$ *and* $\mathbf{u}$ *is a solution of* $\min_{\mathbf{v} \in K} P\pi(\mathbf{v}, \Omega)$.

Proof. This result is proved exactly like Proposition 3.1. $\square$

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